PHYSICAL REVIEW E **66**, 051306 (2002)

Breakdown of self-organized criticality in sandpiles

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(Received 3 July 2002; published 20 November 2002)

We introduce two sandpile models which show the same behavior of real sandpiles, that is, an almost self-organized critical behavior for small systems and a dominance of large avalanches as the system size increases. The systems become fully self-organized critical, with the critical exponents of the Bak, Tank, and Wiesenfeld model [P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987)] as the system parameters are changed, showing that they can make a bridge between the well known theoretical and numerical results and what is observed in real experiments.

DOI: 10.1103/PhysRevE.66.051306 PACS number(s): 45.70.Ht, 05.65.+b, 91.45.Dh

The concept of self-organized criticality (SOC) was introduced by Bak, Tang, and Wiesenfeld (BTW) in 1987 to denote a phenomenon in which out of equilibrium, multidimensional systems drive themselves to a critical state characterized by a power-law distribution of event sizes [1]. Until then, the studies of fractal structures were related to equilibrium systems where this kind of distribution appears only at special parameter values in which a phase transition takes place. In that pioneering work, the concept of SOC was illustrated by a model for sandpiles and since then an enormous amount of research on SOC, both theoretically and experimentally, has been done. Among other phenomena in which SOC has been connected with are earthquakes [2] and evolution [3].

The existence of SOC in an experiment with a quasi-one-dimensional pile of rice was demonstrated by Frette *et al.* [4]. They found that the occurrence of SOC depends on the shape of the rice. Only with sufficient elongated grains, avalanches with a power-law distribution occurred. For more symmetric grains a stretched exponential distribution was seen. Christensen *et al.* [5] introduced a model for the elongated rice pile experiment in which the local critical slope varies randomly between 1 and 2. They found that their model, known as the Oslo rice pile model, reproduced well the experimental results on the quasi-one-dimensional rice pile. In a recent publication [6], we introduced a fully deterministic one-dimensional SOC system, which presents the same qualitative and quantitative behavior of the Oslo system. In other words, they belong to the same universality class.

When one goes to sandpiles with geometry of two dimensions a different picture emerges. That is, the models [1] predict the presence of power-law distributions and the experiments do not display them. The most well known sandpile experiments can be classified into two types: (a) local dropping of sand in the center of the pile [7] and (b) uniform driving, more specifically the rotating drum experiments [8,9]. In type (a) it was found that small systems present scaling almost consistent with SOC, but in large systems another regime with big avalanches belonging to a different distribution appears. In type (b) one sees small avalanches that seem to display a power-law distribution of limited size. These small avalanches are interwoven with big system-wide avalanches which belong to a different distribution. There are studies of avalanches in natural settings [10], which also show similar distribution as the ones observed in the sandpile experiments.

Although SOC has not been observed in sandpile experiments, it is well known that power-law distributions do exist in nature, one of the most well known cases being Gutenberg-Richter law for earthquakes [11] in agreement with SOC. It is clear that there is a missing piece in this puzzle. That is, would there be simple models that would display the observations in real sandpiles, and still present SOC in other parameter regions? To our knowledge, such models are still missing in the literature and it is the aim of this paper to introduce them.

The key ingredient in our models is a nonlinear friction force. Such kind of force has been observed in experiments of sliding rocks [12] and suggested to be present in granular flows [13]. The nonlinear friction force will generate a rich dynamics and destroy SOC via chaotic behavior as the parameters are changed.

Our models are inspired by the model introduced in Ref. [6], and are also governed by a coupled map lattice (that is, the systems characterized by discrete time and continuous values for the space variables). Here we increase the dimensionality and change the drive and the relaxation rules. Having in mind the sandpiles experiments we first introduce a model for the local dropping of sand. We assume a two-dimensional square lattice of linear size L and to each site i,j in the lattice there is associated to it a variable $x_{i,j}$ with $x \in [0,+\infty)$, which is to represent the local slope of the pile. The dynamics of the model is described by the following algorithm.

(1) Start the system by assigning random initial values for the variables $x_{i,j}$, so that they are below a chosen, fixed, threshold x_{th}.

(2) Choose a nearly central site of the lattice and update its slope according to $x_{i,j}=x_{th}$.

(3) Check the slope in each element. If an element i,j has $x_{i,j} \geqslant x_{th}$, update $x_{i,j}$ according to $x'_{i,j}=\phi(x_{i,j}-x_{th})$, where ϕ is a given nonlinear function that has two parameters a and d. Increase the slope in all its nearest neighboring elements according to $x'_{nn}=x_{nn}+\Delta x/4$, where $\Delta x = x_{i,j}-x'_{i,j}$ and nn denotes nearest neighbors.

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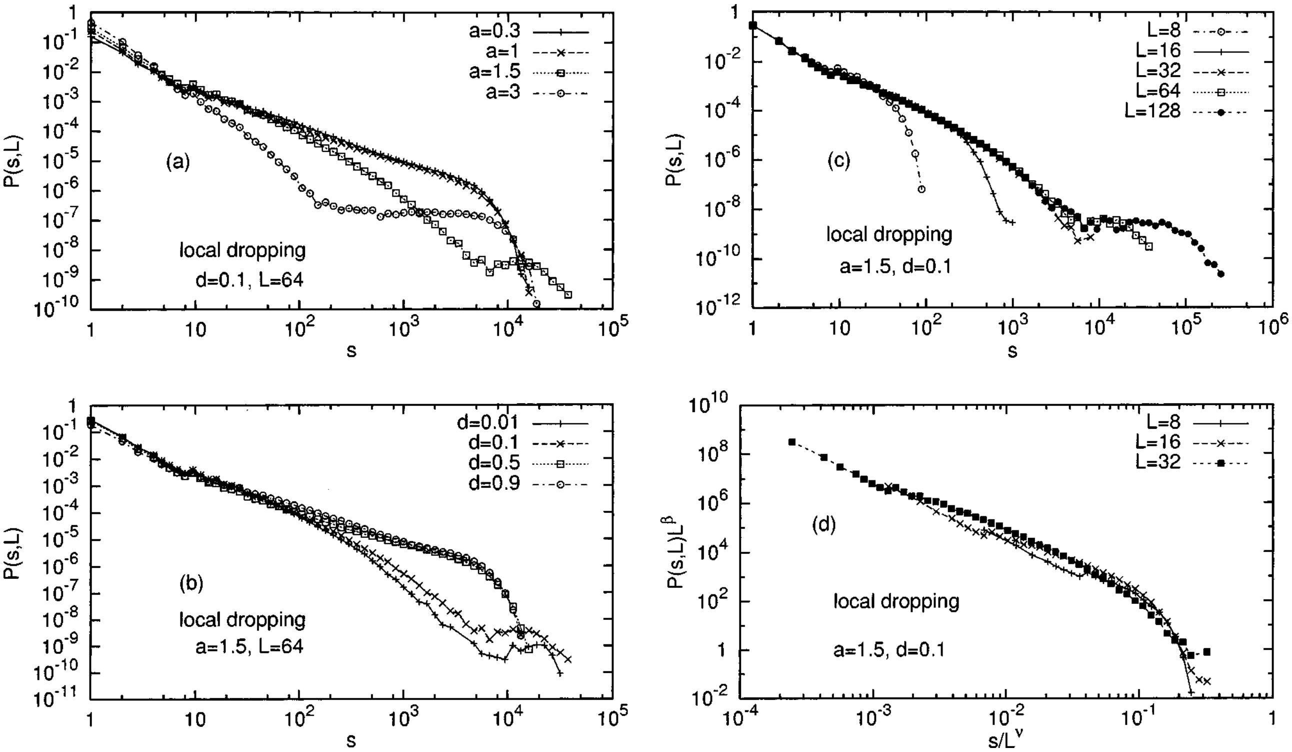

FIG. 1. (a) $P(s,L)$ for the model of local dropping of sand for (a) variable a with $d=0.1$, and (b) variable d and $a=1.5$ with $L=64$. In (c) we vary L and use $a=1.5$ and $d=0.1$, and in (d) we show the fitting using the scaling of Eq. (2) for small systems, with $\beta=6$ and $\nu=3$. We have used 4×10^6 avalanches in all the simulations of this paper.

(4) If $x'_{i,j}<x_{th}$ for all the elements, go to step (2) (the event, or avalanche has finished). Otherwise, go to step (3) (the event is still evolving).

Without losing generality, we can take $x_{th}=1$. In our simulation in step (2) we have chosen the site with $i=j=L/2$. The nonlinear function we use is

$$\phi(x)=\begin{cases}1-d-ax & \text{if } x<(1-d)/a\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

The parameter d would be associated with the minimum drop in energy after an event involving one single element (and also represents a Coulomb-type discontinuity) and a would be the parameter associated with the amount of dynamic friction between the grains. That is, the smaller the a, the larger the friction and smaller the change in the slope of the pile. We have tested several other similar functions and found that the quantitative and qualitative results we show here are robust. In contrast with the one-dimensional case [6], it is not required here that $\phi(x)$ be periodic in order to find the presence of SOC.

We have chosen to evolve the system using parallel dynamics with open boundary conditions. It is beyond the scope of the present paper to study the several possible variations of our models. Further results on these models will be presented in a future publication [14]. The distribution of time duration of the avalanches will also be presented in the future, but our preliminary results show that they are qualitatively similar to the ones for the size distribution.

We display an example of our simulations in Fig. 1, where we show the distribution of events $P(s)$ involving s update steps, that is, the size of the avalanche. The events that involve all the elements of the system have been excluded from our analysis. However, in this model of local dropping of sand we have observed that nothing very distinct will occur if they are also included in the statistics. In (a) we show $P(s)$ for $L=64$, $d=0.1$, and vary a and in (b), we use $a=1.5$ and vary d, keeping $L=64$. We notice the existence of the two regimes. For small d, power-law distributions, that is SOC, appear only when $a\leqslant1$, whereas if d is large, we see SOC even with $a>1$. If d is small and $a>1$ we observe a regime of almost SOC behavior, as in real sandpiles [7], characterized by a region with an apparent scaling for small events, and big events belonging to a different distribution. To illustrate this, we show in Fig. 1(c) simulations with $a=1.5$ and $d=0.1$ and varying L. In small systems $P(s)$ can be fit to a scaling form of the type

$$P(s,L)=L^{-\beta}G(s/L^{\nu}), \tag{2}$$

as shown in Fig. 1(d), where we have used $\beta=6$ and $\nu=3$. The function G is not well fit by a power law, since one can clearly see in the figure that it is curved. We have found [14] that it is consistent with a stretched exponential, as in real sandpiles [15]. The observations of Himalayan avalanches [10] also display the kind of distribution shown in Fig. 1(c) for large systems.

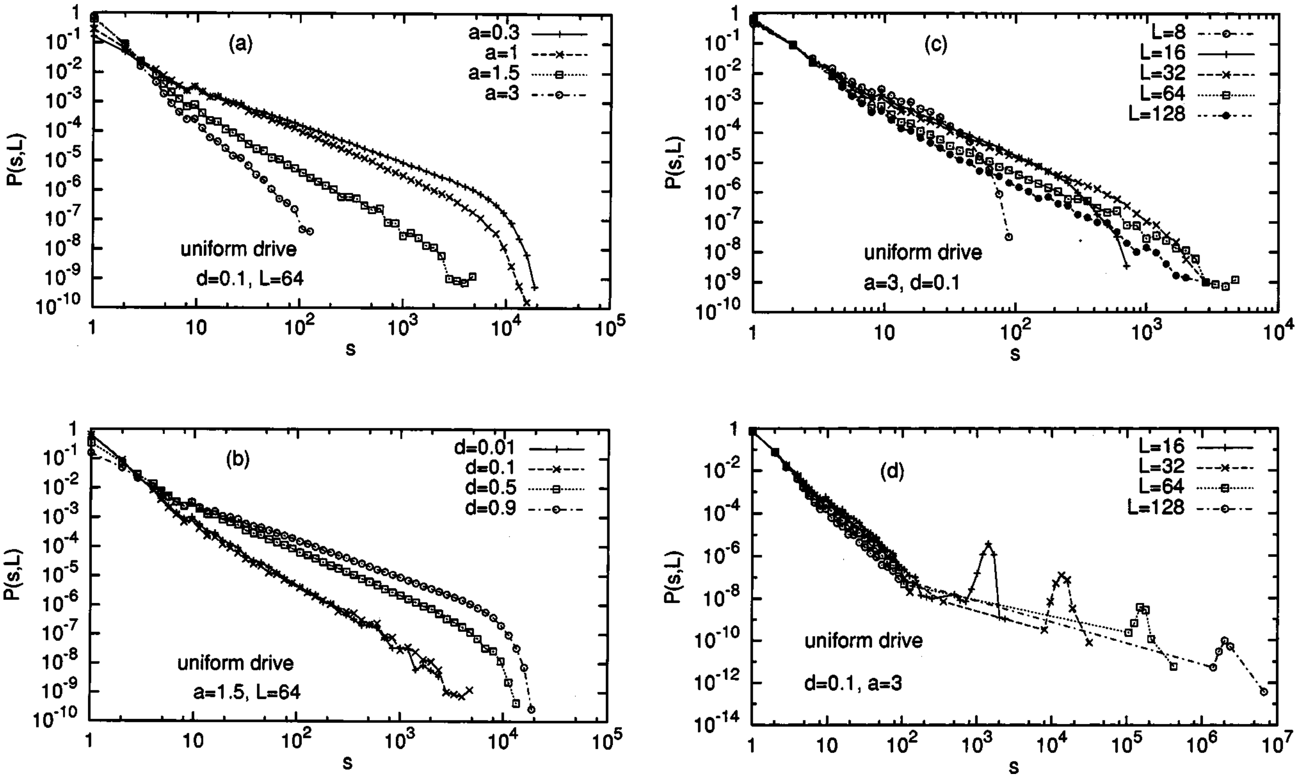

FIG. 2. (a) $P(s,L)$ for the model of uniform drive for (a) variable a with $d=0.1$, and (b) variable d and $a=1.5$ with $L=64$. In (c) we vary the system size and use $a=1.5$ and $d=0.1$. In (d) we use $a=3$ and $d=0.1$ with varying L. The peaks in the distribution are the events involving all the elements of the system.

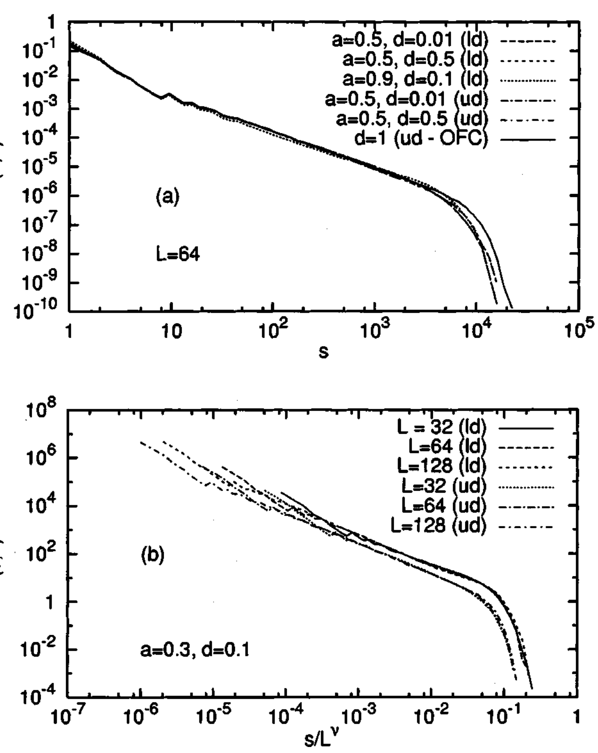

FIG. 3. (a) $P(s,L)$ showing the SOC behavior of our two models when $a \leqslant 1$. The case of the uniform drive is shown when $d=1$, which corresponds to the OFC model. (b) $P(s,L)$ for $a=0.3$ and $d=0.1$ with varying L using the scaling of Eq. (2).

The second model we introduce here is for the rotating drum experiment [8,9], which we call uniform drive since the slope of the pile increases uniformly for all the grains. The algorithm is similar to the one described above, with the exception of step (2), which is now replaced by the following.

(2) Find the element in the lattice that has the largest x, denoted here by x_{max}. Then update all the lattice elements according to $x_{i,j} \mapsto x_{i,j} + x_{th} - x_{max}$.

That is, this model has a dynamics analogous to the OFC model, except for the relaxation rule, which in our case is a piecewise linear function. We show examples of $P(s)$ for this model in Fig. 2. There, in (a) we fix d and vary a, and in (b) we fix a and vary d. In both cases we have used $L=64$ and the events that involve all the elements of the system have been excluded. Distinctly from the model of local dropping, it seems here that there is a power-law distribution for any parameter value. However, the behavior is not exactly SOC. We have found that SOC is only seen if $a \leqslant 1$ or d is large enough, as in the case of the local dropping. When $a > 1$ and d is smaller than a given value, we see a SOC-like behavior only for small values of L. As L grows, there is a transition to a different behavior, in which the larger the system, the smaller the power-law region is, as Fig. 2(c) shows.

System-wide avalanches have been reported in the rotating drum experiment [8,9], which belong to a different distribution than the one of the small avalanches. This is exactly what we see in this model for $a > 1$ and small d. In Fig. 2(d) we show all the events of the system including the ones that involve all the elements (for $a=3$, $d=0.1$, and varying L). We see that the scaling region does not get bigger as the system size increases, and a peak related to the events involving all the elements is seen. We have found that the intermediate size events ($10 \lesssim s \lesssim 100$) can be fit by a scaling of the type $P(s,L)=L^{-\beta}G(s/L^{\nu})$ with $\beta=1$ and $\nu=0$. As in the rotating drum experiment [9,15] the function G in this case is closer to a power law than in the case of the experiment with local dropping of sand.

In Fig. 3(a) we show the SOC regime observed when $a \leqslant 1$, for both the local dropping and uniform drive models. We see that for given a and d the slope of the power-law distribution seems to be the same for both models, that is, the local dropping of sand and uniform drive, but the slope varies slightly with a, as seen also in Figs. 1 and 2. Consequently, the universality class of these models vary with the parameters. In Fig. 3(b) we show the scaling given by Eq. (2) with $\beta=3.55$ and $\nu=2.70$ for the local dropping (LD) and $\beta=3.55$ $\nu=2.85$ for the uniform drive (UD). Our second model is a generalized version of two well known SOC models: the earthquake model of Olami, Feder, and Christensen (OFC) [2], which is recovered when $a=0$ or $d=1$ and the sandpile model introduced by BTW [1], which is recovered if $a=0$, $d=4$, and x is transformed into an integer variable. In fact, our second model in this limit gives $P(s) \sim s^{-1.25}$, as those two ones.

We next investigate what would happen if the relaxation function is just a random number generator. In other words, instead of using ϕ in step 3 of the above algorithm we now use $x'_{i,j}=\rho$, where ρ is a random number uniformly distributed in the interval $[0,\ 1-d]$. We have found SOC for any $d \in (0,\ 1]$ with the same exponents as the BTW model. This is displayed in Fig. 4(a) where we show the size distribution for $d=0.01$ in the cases of LD and UD. Therefore, nonlinearities in ϕ and consequently nonergodicity are necessary for the SOC behavior to be destroyed in these models. In that figure, we also show the case in which $x'_{i,j}=0$, which corresponds to $d=1$. In this limit we recover the OFC [2] model for the case of uniform drive.

The reason why $a=1$ determines a special boundary, in which SOC may or may not be present, is due to the fact that it marks the boundary in which strong instabilities exist. In fact, when $a>1$ one sees the exponential divergence of trajectories, the hallmark of chaotic behavior [16], which start with almost the same initial conditions. That is, if we consider two copies of one of our models, copy 1 and copy 2, with all the elements having the same x except that in copy 1 the element $x^1_{i,j}=x^*$, whereas in copy 2 $x^2_{i,j}=x^*+\Delta$. It is not difficult to find that after one iteration by ϕ, the separation of the two elements instead of Δ will be $a\Delta$. So, if $a>1$ the separation increases, whereas if $a<1$ the separation decreases. In our models the transition from SOC to non SOC is continuous in d and discontinuous in a.

The next natural questions are: what causes the appearance of another regime when the system size increases (for $a>1$ and small d) and what marks the boundary between the two regimes of the distribution? We verified that when $d \to 0$ the distribution of avalanches can be decomposed into two components: one for the events that do not probe the nonlinear regime [that is, the flat part of the friction force given by Eq. (2)] and another for the events that do. It turns out that the events that do not probe the nonlinear regime are the ones in the scaling regime of almost SOC, whereas the events that probe the nonlinear regime belong to a different distribution. To illustrate this, we show in Fig. 4(b) the distribution of events that are only in the linear regime and the ones that are not for the case of $d=0.01$ and $a=1.5$, with $L=16$ and $L=64$. Therefore, the existence of nonlinearities in the friction force generates the distinct kind of distribution we observe, which is also displayed by real sandpiles. For the case in which d is not so small, similar results are observed, but it seems that in this case a third regime appears, which is currently under investigation. Studies of the relationship between our models and analytical studies, such as mean field approach [17] are currently in progress.

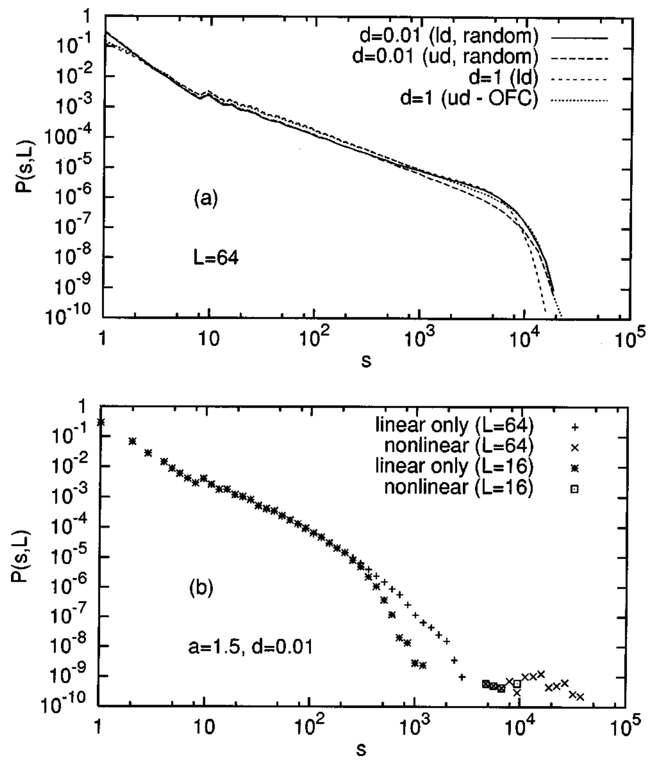

FIG. 4. (a) $P(s,L)$ for the cases in which after relaxation the variable x is assigned a random number uniformly distributed between 0 and 0.99 for local dropping and for uniform drive. We also show the cases in which x is relaxed to 0 after an event, which is the same as having $d=1$. In (b) we show the distribution of events for $a=1.5$, $d=0.01$, with $L=16$ and $L=64$ for the events that attain only the linear regime of the friction force, and for the ones that do not.

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